Electrical and Electronics Engineering
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Master Semester 2

Course
Smart grids technologies

Numerical solution of the load flow problem formulated via the nodal analysis

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Introduction

$$[\mathbf{I}]=[\mathbf{Y}][\mathbf{V}]$$

In a network with s nodes, the s complex voltages and the s complex node currents are linked by s equations with complex variables and coefficients representing the **internal network constraints**.

The $2s$ **complex voltages** and **currents** are equivalent to $4s$ **real variables**. On the other hand, we have $2s$ **linear equations** with real variables and coefficients, which are obtained by separating the real and imaginary parts (or the modules and the arguments).

Therefore, from the $4s$ real variables, $2s$ can be fixed arbitrarily and the remaining $2s$ are calculated by solving the system of equations of the network.

When the system is solved, and therefore all voltages and currents are known, we can calculate:

- P and Q injected or absorbed by the nodes
- branches powers and currents
- network losses (both active and reactive, corresponding to the balance between powers injected and absorbed by the nodes).

If the network operating conditions can be represented **imposing as external constraints only voltage/current amplitudes and phases**, **the power flows can be calculated by solving a simple system of linear equations.**

Introduction

In practice, the operating conditions imposed on the networks (***external constraints***) are expressed by **fixing other parameters**. This implies that the system of equations to be solved becomes non-linear. In particular:

In the **load buses** P and Q demands are (usually) fixed ($\boldsymbol{P_i}$* and $\boldsymbol{Q_i}$*).

In practice, **it is inappropriate to represent the various loads by constant admittances** (i.e., asynchronous motors absorb active power almost independently of the voltage, with variations in the range of +10%; gas-discharge lamps and incandescent lamps, even if they absorb power that varies with the voltage, do not follow a quadratic law).

The dependency on the voltage of the active/reactive powers absorbed by the loads can be expressed by exponential models where the values of the exponent coefficients depend to the nature of the load and, in some cases, it can also be set = 0.

$$P = P_0 \left(\frac{V}{V_0} \right)^{\alpha_p}$$

$$Q = Q_0 \left(\frac{V}{V_0} \right)^{\alpha_q}$$

Introduction

For the **generator buses**, it is convenient to fix the P that is injected from them into the grid ($\boldsymbol{P_i}$*) and the amplitude of voltage V ($\boldsymbol{V_i}$*).

- For power transmission networks, the value $\boldsymbol{P_i}$* that each power plant is asked to provide (at a given time step) is usually selected in accordance to their dispatching.
- Fixing the $\boldsymbol{V_i}$*, rather than the Q, is convenient for the following reasons:
  1. fixing the voltage (typically at a value between the rated voltage V_n and $1.1V_n$ according to the location of the power plants and the distance with respect to the loads), means that we set the voltage in the network key points (often scattered throughout the network). So the solution of the equations provides a solution acceptable for the network operation. This choice also facilitates the convergence of the iterative procedure for the solution of the LF equations (see next).
  2. The Q of the generators can vary between the max-limit (i.e., generators over excitation) and the min-limit (i.e., generators under excitation) by simply varying the their excitation current. Therefore, it is convenient to accept to operate each power plant with the Q that is provided by the calculation and which allows to obtain the predetermined voltages.

Introduction

We have justified that both for loads and generators, it is convenient to fix the P.

It should be noted, however, that it is not possible to assign arbitrary values of P at all nodes because this would be equal to **arbitrarily setting the network losses, which is clearly absurd**. In fact, the losses are not known initially, but are calculated together with the power flows, after having solved the LF equations.

It is therefore allowable to arbitrarily set no more than (s-1) active powers.

Consequently, for one of the nodes, that can be chosen to coincide in the numbering with the s-th node, the amplitude and the phase of the voltage are fixed. This node is called **slack bus**, as the active power, for this node, is equal to the balance between the active powers of generators/loads and the power losses.

As slack bus we can choose a generator where a significant power is installed. In this node **the phase of the voltage is fixed to zero**; this is equivalent to measuring the phases of the other (s-1) node voltages using as a **reference the slack bus voltage phasor**.

Introduction

Summary of the parameters that are imposed and the ones that need to be determined for the various types of buses

Type of bus	Imposed parameters (in total $2s$)		Parameters to be determined (in total $2s$)	
Generator buses	P_g	V_g	Q_g	$\arg(V_g)$
Load buses	P_c	Q_c	V_c	$\arg(V_c)$
Slack bus	V_n	$\arg(V_n) = 0$	P_n	Q_n

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Cartesian Coordinates Formulation

We will use the following notations:

$\bar{V}_i = V'_i + jV''_i$ voltage of the i-th node;

$\bar{V}_\ell = V'_\ell + jV''_\ell$ voltage of the ℓ-th node;

$\bar{Y}_{i\ell} = G_{i\ell} + jB_{i\ell}$ $i\ell$ element of the admittance matrix $[\mathbf{Y}]$;

The complex power injected, or absorbed, from the i-th node can be written as:

$$\bar{S}_i = P_i + jQ_i = \bar{V}_i \underline{I}_i$$

replacing the expression that gives the complex current inserted or extracted from the node i we get:

$$\bar{S}_i = \bar{V}_i \sum_{\ell=1}^{s} \underline{Y}_{i\ell} \underline{V}_\ell = \left(V'_i + jV''_i\right) \sum_{\ell=1}^{s} \left(G_{i\ell} - jB_{i\ell}\right)\left(V'_\ell - jV''_\ell\right)$$

Cartesian Coordinates Formulation

Then, the injected active and reactive powers of the i-th node are:

$$\begin{cases} P_i = V_i^{'} \sum_{\ell=1}^{s} \left(G_{i\ell} V_\ell^{'} - B_{i\ell} V_\ell^{''} \right) + V_i^{''} \sum_{\ell=1}^{s} \left(B_{i\ell} V_\ell^{'} + G_{i\ell} V_\ell^{''} \right) \\ Q_i = -V_i^{'} \sum_{\ell=1}^{s} \left(B_{i\ell} V_\ell^{'} + G_{i\ell} V_\ell^{''} \right) + V_i^{''} \sum_{\ell=1}^{s} \left(G_{i\ell} V_\ell^{'} - B_{i\ell} V_\ell^{''} \right) \end{cases}$$

The module (squared) of the voltage at the i-th node are:

$$V_i^2 = V_i^{'2} + V_i^{''2}$$

Cartesian Coordinates Formulation

The entire system of equations in cartesian coordinates assumes the following form:

$$0 = V_i''$$

i=s for the unique slack bus

$$V_i^{*2} = V_i'^2 + V_i''^2$$

i=1,2,...,g and i=s, for the g generator buses + the slack bus

$$P_i^* = V_i' \sum_{\ell=1}^{s} \left(G_{i\ell} V_\ell' - B_{i\ell} V_\ell'' \right) + V_i'' \sum_{\ell=1}^{s} \left(B_{i\ell} V_\ell' + G_{i\ell} V_\ell'' \right)$$

i=1,2,...,g+u, for the g generator buses + u load buses

$$Q_i^* = -V_i' \sum_{\ell=1}^{s} \left(B_{i\ell} V_\ell' + G_{i\ell} V_\ell'' \right) + V_i'' \sum_{\ell=1}^{s} \left(G_{i\ell} V_\ell' - B_{i\ell} V_\ell'' \right)$$

i=g+1,..., g+u for the load buses

The number of equations is: $1+(g+1)+(g+u)+u=2(g+u+1)=2s$.

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Polar Coordinates Formulation

By indicating with φ_i, θ_i respectively the arguments of the current and voltage of node i and with $\gamma_{i\ell}$ the argument of the admittance $\bar{Y}_{i\ell}$, we can write

$\bar{V}_i = V_i e^{j\vartheta_i}$ voltage at the i-th node;

$\bar{I}_i = I_i e^{j\phi_i}$ current at the i-th node;

$\bar{Y}_{i\ell} = Y_{i\ell} e^{j\gamma_{i\ell}}$ element $i\ell$ of the admittance matrix $[\mathbf{Y}]$;

The complex power at the i-th node can be written as:

$$\bar{S}_i = P_i + jQ_i = \bar{V}_i \underline{I}_i$$

Using again the network admittance matrix to express the injected node current, we obtain:

$$\bar{S}_i = \bar{V}_i \sum_{\ell=1}^{s} \underline{Y}_{i\ell} \underline{V}_\ell = \sum_{\ell=1}^{s} \bar{V}_i \underline{Y}_{i\ell} \underline{V}_\ell = \sum_{\ell=1}^{s} V_i V_\ell Y_{i\ell} e^{j(\vartheta_i - \vartheta_\ell - \gamma_{i\ell})}$$

Polar Coordinates Formulation

Then, the active and reactive powers at the i-th node will be:

$$\begin{cases} P_i = \sum_{\ell=1}^{s} V_i V_\ell Y_{i\ell} \, cos\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \\ Q_i = \sum_{\ell=1}^{s} V_i V_\ell Y_{i\ell} \sin\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \end{cases} \qquad \text{(LF.1)}$$

Polar Coordinates Formulation

The system of equations for the solution of the load flow problem in polar coordinates assumes therefore the following form:

$$0 = \vartheta_i$$

$i=s$ for the unique slack bus

$$V_i^* = V_i$$

$i=1,2,\ldots,g$ and $i=s$, for the g generator buses + the slack bus

$$P_i^* = V_i \sum_{\ell=1}^{s} V_\ell Y_{i\ell} \cos\left(\vartheta_{i\ell} - \gamma_{i\ell}\right)$$

$i=1,2,\ldots,g+u$, for the g generator buses + the u load buses

$$Q_i^* = V_i \sum_{\ell=1}^{s} V_\ell Y_{i\ell} \sin\left(\vartheta_{i\ell} - \gamma_{i\ell}\right)$$

$i=g+1,\ldots,g+u$, for the load buses

The number of equations is: $1+(g+1)+(g+u)+u=2s$.

Polar Coordinates Formulation

In this case, the first $g+2$ equations (generator buses and slack bus) directly imposes the value of the external constraints, so the number of equations needed in the formulation in polar coordinates ($g+2u$) is lower than the one in Cartesian coordinates.

This does not necessarily imply a reduction of the computation time. In fact, using polar coordinates, it is necessary to calculate trigonometric functions sin and cos.

Polar Coordinates Formulation

Formulation in polar coordinates for the voltage and cartesian for the admittances:

$$0 = \vartheta''_s$$

i=s for the unique slack bus

$$V_i^* = V_i$$

i=1,2,...,g and i=s, for the g generator buses + the slack bus

$$P_i^* = V_i \sum_{\ell=1}^{s} V_\ell \left(G_{i\ell} \cos \vartheta_{i\ell} + B_{i\ell} \sin \vartheta_{i\ell} \right)$$

i=1,2,...,g+u, for the g generator buses + the u load buses

$$Q_i^* = V_i \sum_{\ell=1}^{s} V_\ell \left(G_{i\ell} \sin \vartheta_{i\ell} - B_{i\ell} \cos \vartheta_{i\ell} \right)$$

i=g+1,..., g+u, for the load buses

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Line Power Flows

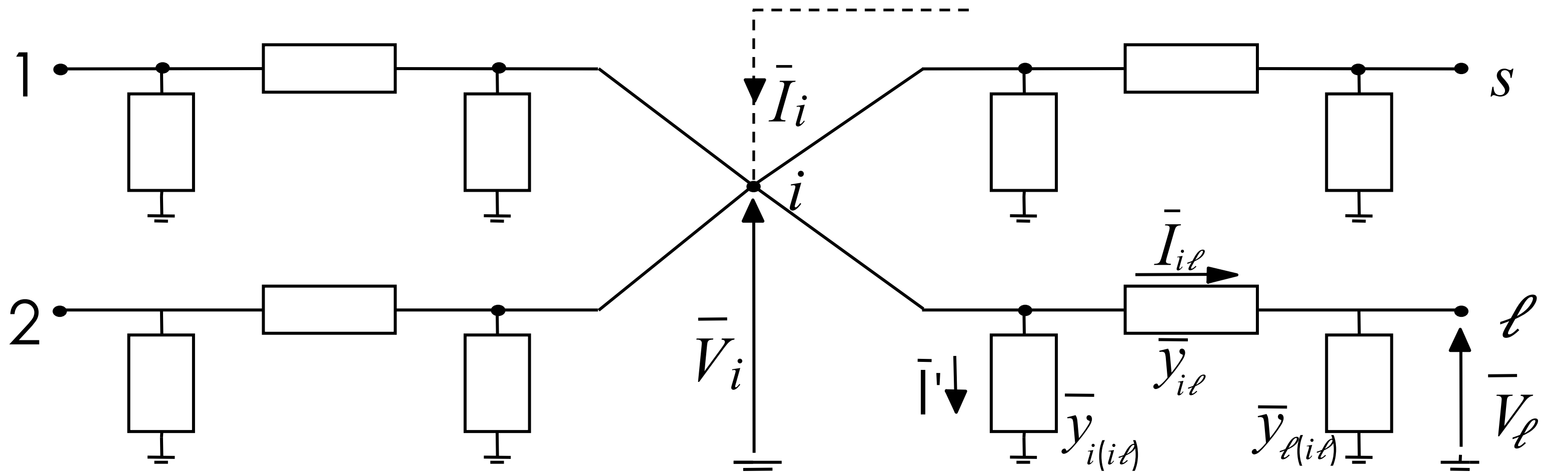

Fig.LF_2. Power flow in the line *ih*.

$$\bar{S}_{i\ell} = P_{i\ell} + jQ_{i\ell} = \bar{V}_i\left(\underline{V}_i - \underline{V}_\ell\right)\underline{y}_{i\ell} + V_i^2\,\underline{y}_{i(i\ell)} \qquad \text{(LF.2)}$$

Polar

$$\bar{S}_{i\ell} = \bar{V}_i\left(\underline{V}_i - \underline{V}_\ell\right)\underline{y}_{i\ell} + V_i^2\,\underline{y}_{i(i\ell)}$$

$$= V_i e^{j\vartheta_i}\left(V_i e^{-j\vartheta_i} - V_\ell e^{-j\vartheta_\ell}\right) y_{i\ell} e^{-j\gamma_{i\ell}} + V_i^2 y_{i(i\ell)} e^{-j\gamma_i} \qquad \text{(LF.3)}$$

$$= V_i^2 y_{i\ell} e^{-j\gamma_{i\ell}} - V_i V_\ell y_{i\ell} e^{j\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right)} + V_i^2 y_{i(i\ell)} e^{-j\gamma_i}$$

Having defined with γ_i the argument of the admittance $y_i(i\ell)$ and with $\boldsymbol{\vartheta}_i$ and $\boldsymbol{\vartheta}_l$ the arguments of the node voltage phasors.

Line Power Flows

Formulation in cartesian coordinates:

$$P_{i\ell} = \left(g_{i\ell} + g_{i(i\ell)}\right)\left(V_i'^2 + V_\ell''^2\right) - g_{i\ell}\left(V_i'V_\ell' + V_i''V_\ell''\right) + b_{i\ell}\left(V_i'V_\ell'' - V_\ell'V_i''\right)$$

$$Q_{i\ell} = -\left(b_{i\ell} + b_{i(i\ell)}\right)\left(V_i'^2 + V_\ell''^2\right) + g_{i\ell}\left(V_i'V_\ell'' - V_\ell'V_i''\right) + b_{i\ell}\left(V_i'V_\ell' + V_i''V_\ell''\right)$$

Formulation in polar coordinates for the voltage and cartesian for the admittances:

$$P_{i\ell} = V_i^2\left(g_{i\ell} + g_{i(i\ell)}\right) - V_iV_\ell\left(g_{i\ell}\cos\vartheta_{i\ell} + b_{i\ell}\sin\vartheta_{i\ell}\right)$$

$$Q_{i\ell} = -V_i^2\left(b_{i\ell} + b_{i(i\ell)}\right) - V_iV_\ell\left(g_{i\ell}\sin\vartheta_{i\ell} - b_{i\ell}\cos\vartheta_{i\ell}\right)$$

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Numerical solution via the NR method

Since the equations that link the unknown network variables with those that are known are non-linear, they must be resolved by using iterative numerical procedures: starting from a **reasonable initial profile** (for instance, **all the unknown amplitudes set equal to 1 per unit or to the value of the slack bus, all the unknown phases set equal to the phase of the slack bus**), they are progressively updated until the convergence, according to one of the procedures provided by the numerical analysis. The selection of the initial profiles is generally such that, if the process converges, it can guarantee that **the solution has a physical meaning (i.e., voltages within bounds and line currents below the ampacity limits)**. The most common iterative methods are based on the description of the network in terms of the nodal admittance matrix, although there are also different procedures.

The most common convergence criterion, as it will be explained below, refers to the **control of the active and reactive residuals, equal to the differences between the corresponding fixed and calculated powers.**

Numerical solution via the NR method

There is a given **differentiable**, non-linear, function of x, $f(x)$. We want to determine the value of x, let's call it x^*, so that the function $f(x)$ has a fixed value y^*, namely the value of x that satisfies the equation:

$$f(x^*) = y^* \qquad \text{(LF.4)}$$

We indicate with $x^{(0)}$ a value close to x^*, and their relative difference is:

$$\Delta x^{(0)} = x^* - x^{(0)} \qquad \text{(LF.5)}$$

If we develop $f(x)$ in Taylor series at the point $x^{(0)}$ we get:

$$f(x^{(0)} + \Delta x^{(0)}) = f(x^{(0)}) + \left(\frac{df}{dx}\right)^{(0)} \Delta x^{(0)} + \left(\frac{d^2 f}{dx^2}\right)^{(0)} \frac{\left(\Delta x^{(0)}\right)^2}{2!} + \dots \qquad \text{(LF.6)}$$

For the values of $x^{(0)}$ that are sufficiently close to x^*, we can write (i.e., terms with order higher than the first are neglected):

$$f(x^*) \cong f(x^{(0)}) + \left(\frac{df}{dx}\right)^{(0)} \Delta x^{(0)} \qquad \text{(LF.7)}$$

Numerical solution via the NR method

The equation (LF.4) can be re-written as:

$$f(x^{(0)}) + \left(\frac{df}{dx}\right)^{(0)} \Delta x^{(0)} \cong y^* \qquad \text{(LF.8)}$$

From (LF.8) we can obtain the value of $\Delta x^{(0)}$ which, however, will not be equal to the one expressed in (LF.5), because of the introduced approximation. Therefore, we will call it $\Delta x^{(1)}$.

$$\Delta x^{(1)} = \frac{y^* - f(x^{(0)})}{\left(\frac{df}{dx}\right)^{(0)}} = \frac{\Delta y^{(0)}}{\left(\frac{df}{dx}\right)^{(0)}} \qquad \text{(LF.9)}$$

where we set:

$$\Delta y^{(0)} = y^* - f(x^{(0)}) \qquad \text{(LF.10)}$$

Hence, if in the initial equation we add the value of $\Delta x^{(1)}$ derived by (LF.9), we obtain a new value of x, let's call it $x^{(1)}$, that is closer to the actual solution of the equation:

Numerical solution via the NR method

$$x^{(1)} = x^{(0)} + \Delta x^{(1)} = \begin{cases} x^{(0)} + \dfrac{\Delta y^{(0)}}{\left(\dfrac{df}{dx}\right)^{(0)}} & ,\text{if} \quad y^* \neq 0 \\ x^{(0)} - \dfrac{f(x^{(0)})}{\left(\dfrac{df}{dx}\right)^{(0)}} & ,\text{if} \quad y^* = 0 \end{cases} \qquad \text{(LF.11)}$$

If we generalize in the $\nu + 1$ -*th* iteration, we get:

$$x^{(\nu+1)} = x^{(\nu)} + \Delta x^{(\nu+1)} = \begin{cases} x^{(\nu)} + \dfrac{\Delta y^{(\nu)}}{\left(\dfrac{df}{dx}\right)^{(\nu)}} & ,\text{if} \quad y^* \neq 0 \\ x^{(\nu)} - \dfrac{f(x^{(\nu)})}{\left(\dfrac{df}{dx}\right)^{(\nu)}} & ,\text{if} \quad y^* = 0 \end{cases} \qquad \text{(LF.12)}$$

Numerical solution via the NR method

where:

$$\Delta x^{(\nu+1)} = x^{(\nu+1)} - x^{(\nu)}$$
$$\Delta y^{(\nu)} = y^{*} - f\left(x^{(\nu)}\right) \qquad \text{(LF.13)}$$

Then, we obtain the following expression of (LF.12):

$$\left(\frac{df}{dx}\right)^{(\nu)} \Delta x^{(\nu+1)} = \begin{cases} \Delta y^{(\nu)} & ,\text{if} \quad y^{*} \neq 0 \\ -f\left(x^{(\nu)}\right) & ,\text{if} \quad y^{*} = 0 \end{cases} \qquad \text{(LF.14)}$$

This relation is the **main iterative equation of Newton-Raphson**.

The graphical representation of the process, in the case that $y^{*}=0$, is shown in Fig. LF_3 (which, among others, also justifies the classification of the Newton-Raphson method as **the method of the tangents**).

Numerical solution via the NR method

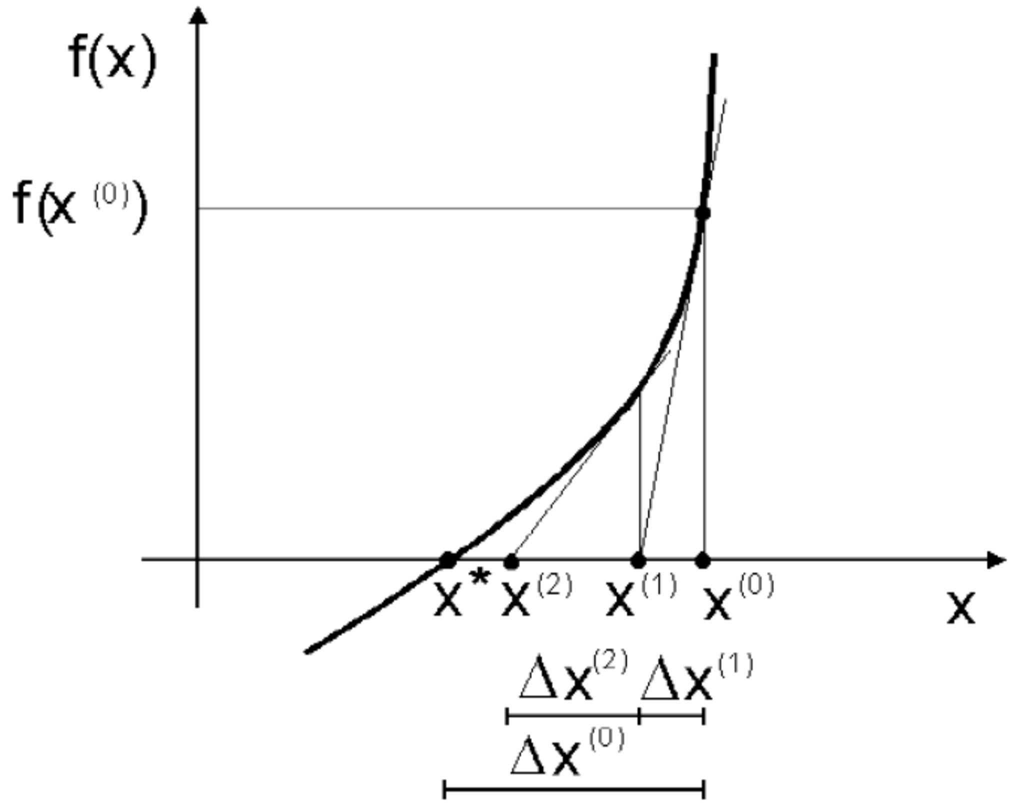


Fig. LF_3. Graphical interpretation of the Newton-Raphson equation (case $y^*=0$).

The consecutive approximations stop when one of the two following conditions are satisfied:

$$\Delta x^{(\nu+1)} < \varepsilon \tag{LF.15}$$

Numerical solution via the NR method

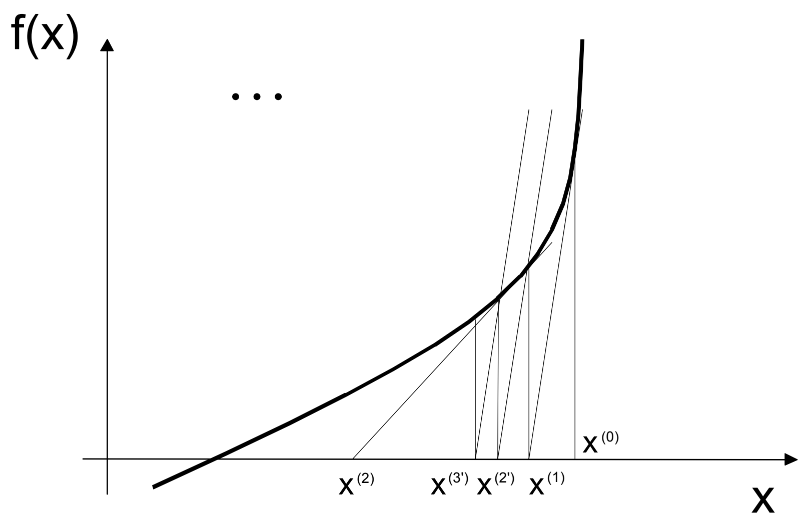
or:

$$\begin{cases} f\left(x^{(\nu+1)}\right) < \varepsilon & ,\text{if} \quad y^* = 0 \\ \Delta y^{(\nu+1)} < \varepsilon & ,\text{if} \quad y^* \neq 0 \end{cases} \qquad (\text{LF.16})$$

where ε is called ***tolerance***.

The parameter $f\left(x^{(\nu+1)}\right)$ (or $\Delta y^{(\nu+1)}$) tends to become smaller (limit to zero) as progressively the process converges to the desired solution: this carachteristic justifies the term "***residual***".

In the proposed procedure the calculation can be simplified with the following modification: when the derivative in $x^{(0)}$ is computed and the new value $x^{(1)}$ is determined, it is possible – without recalculating the new value that it assumes in $x^{(1)}$ – to use again the same one to determine the next value $x^{(2')}$. A process like this, as it is shown in Fig. (LF_4), takes more iterations, but the necessary time to carry out each of them is significantly reduced since it is no more needed to recalculate the derivative (which can be heavy, especially for functions that have more variables).



Numerical solution via the NR method

We conside the following system of s non-linear equations with s unknown quantities:

$$\begin{cases} f_1(x_1, x_2, \dots, x_s) = y_1^* \\ f_2(x_1, x_2, \dots, x_s) = y_2^* \\ \dots \\ f_s(x_1, x_2, \dots, x_s) = y_s^* \end{cases} \tag{LF.17}$$

If we linearize the equations in the same way as we did for the functions of one variable (namely, by truncating the development in Taylor series of the vectorial function in the 1st-order terms), in the $\nu+1$-th iteration we get:

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_s} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_s} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_s}{\partial x_1} & \frac{\partial f_s}{\partial x_2} & \dots & \frac{\partial f_s}{\partial x_s} \end{bmatrix}^{(\nu)} \times \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \dots\dots \\ \Delta x_s \end{bmatrix}^{(\nu+1)} = \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \dots\dots \\ \Delta y_s \end{bmatrix}^{(\nu)} \tag{LF.18}$$

Numerical solution via the NR method

Or, in a compact matrix formulation:

$$[J]^{(\nu)} \times [\Delta x]^{(\nu+1)} = [\Delta y]^{(\nu)} \qquad \text{(LF.19)}$$

having set, as in the case of one variable:

$$\Delta x_i^{(\nu+1)} = x_i^{(\nu+1)} - x_i^{(\nu)} \qquad \text{(LF.20)}$$

and

$$\Delta y_i^{(\nu)} = y_i * - f_i\left(x_1^{(\nu)}, x_2^{(\nu)}, \ldots, x_s^{(\nu)}\right) \qquad \text{(LF.21)}$$

where

$$[J] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_s} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_s} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_s}{\partial x_1} & \frac{\partial f_s}{\partial x_2} & \cdots & \frac{\partial f_s}{\partial x_s} \end{bmatrix} \qquad \text{(LF.22)}$$

Numerical solution via the NR method

The (LF.18) is the iterative equation of Newton-Raphson in the case of a system of s linear equations of s variables. Its inversion allows to determine the updated value of the s-tuple that approximates the solution of the system in the $\nu+1$-th iteration. The new s-tuple is:

$$\begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_s \end{bmatrix}^{(\nu+1)} = \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_s \end{bmatrix}^{(\nu)} + \begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \dots\dots \\ \Delta x_s \end{bmatrix}^{(\nu+1)} \qquad \text{(LF.23)}$$

where, based on (LF.18)

$$\begin{bmatrix} \Delta x_1 \\ \Delta x_2 \\ \dots\dots \\ \Delta x_s \end{bmatrix}^{(\nu+1)} = \left(\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_s} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_s} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_s}{\partial x_1} & \frac{\partial f_s}{\partial x_2} & \cdots & \frac{\partial f_s}{\partial x_s} \end{bmatrix}^{(\nu)} \right)^{-1} \times \begin{bmatrix} \Delta y_1 \\ \Delta y_2 \\ \dots\dots \\ \Delta y_s \end{bmatrix}^{(\nu)} \qquad \text{(LF.24)}$$

Numerical solution via the NR method

In compact form:

$$[x]^{(\nu+1)} = [x]^{(\nu)} = [\Delta x]^{(\nu+1)} \qquad \text{(LF.25)}$$

where

$$[\Delta x]^{(\nu+1)} = \left([J]^{(\nu)}\right)^{-1} \times [\Delta y]^{(\nu)} \qquad \text{(LF.26)}$$

As in the case of one variable only, in order to solve the system of (LF.18), we assume an s-tuple of values ($x_1^{(0)}$, $x_2^{(0)}$,…, $x_s^{(0)}$). We calculate the Jacobian for this set and then, using (LF.21) and (LF.22), we determine the s-tuple ($x_1^{(1)}$, $x_2^{(1)}$,…, $x_s^{(1)}$). At this moment we recalculate the Jacobian in the new determined point and we determine the next point, until the desired accuracy is achieved.

Numerical solution via the NR method

Application of the Newton-Raphson method in the Loadflow (Cartesian):

a) Load buses:

$$\begin{bmatrix} \dfrac{\partial P}{\partial V'} & \dfrac{\partial P}{\partial V''} \\ \dfrac{\partial Q}{\partial V'} & \dfrac{\partial Q}{\partial V''} \end{bmatrix}^{(\nu)} \times \begin{bmatrix} \Delta V' \\ \Delta V'' \end{bmatrix}^{(\nu+1)} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}^{(\nu)} \qquad \text{(LF.27)}$$

where

$$\Delta P_i^{(\nu)} = P_i^* - P_i^{(\nu)} \qquad \text{(LF.28)}$$

$$\Delta Q_i^{(\nu)} = Q_i^* - Q_i^{(\nu)} \qquad \text{(LF.29)}$$

$$\left(\Delta V'\right)_i^{(\nu+1)} = \left(V'\right)_i^{(\nu+1)} - \left(V'\right)_i^{(\nu)} \qquad \text{(LF.30)}$$

$$\left(\Delta V''\right)_i^{(\nu+1)} = \left(V''\right)_i^{(\nu+1)} - \left(V''\right)_i^{(\nu)} \qquad \text{(LF.31)}$$

where the expressions of P_i and Q_i are given here: and the Jacobian is a 2*u* x 2*u* matrix. The derivatives that form the Jacobian are equal to:

Numerical solution via the NR method

$$J_{PR}:\begin{cases}\dfrac{\partial P_i}{\partial V_\ell'} = G_{i\ell}V_i' + B_{i\ell}V_i'' \\[2ex] \dfrac{\partial P_i}{\partial V_i'} = 2G_{ii}V_i' + \sum\limits_{\substack{\ell=1\\ \ell\neq i}}^{s}\left(G_{i\ell}V_\ell' - B_{i\ell}V_\ell''\right)\end{cases} \qquad (\text{LF.32})$$

$$J_{PX}:\begin{cases}\dfrac{\partial P_i}{\partial V_\ell''} = -B_{i\ell}V_i' + G_{i\ell}V_i'' \\[2ex] \dfrac{\partial P_i}{\partial V_i''} = 2G_{ii}V_i'' + \sum\limits_{\substack{\ell=1\\ \ell\neq i}}^{s}\left(B_{i\ell}V_\ell' + G_{i\ell}V_\ell''\right)\end{cases} \qquad (\text{LF.33})$$

Numerical solution via the NR method

$$J_{QR}:\begin{cases}\dfrac{\partial Q_i}{\partial V_\ell^{'}} = -B_{i\ell}V_i^{'} + G_{i\ell}V_i^{''} \\ \dfrac{\partial Q_i}{\partial V_i^{'}} = -2B_{ii}V_i^{'} - \sum_{\substack{\ell=1 \\ \ell\neq i}}^{s}\left(B_{i\ell}V_\ell^{'} + G_{i\ell}V_\ell^{''}\right)\end{cases} \qquad (LF.34)$$

$$J_{QX}:\begin{cases}\dfrac{\partial Q_i}{\partial V_\ell^{''}} = -G_{i\ell}V_i^{'} - B_{i\ell}V_i^{''} \\ \dfrac{\partial Q_i}{\partial V_i^{''}} = -2B_{ii}V_i^{''} + \sum_{\substack{\ell=1 \\ \ell\neq i}}^{s}\left(G_{i\ell}V_\ell^{'} - B_{i\ell}V_\ell^{''}\right)\end{cases} \qquad (LF.35)$$

Numerical solution via the NR method

b) Generator buses:

$$\left[\begin{array}{c:c} \dfrac{\partial P}{\partial V'} & \dfrac{\partial P}{\partial V''} \\ \hdashline \dfrac{\partial V^2}{\partial V'} & \dfrac{\partial V^2}{\partial V''} \end{array}\right]^{(\nu)} \times \left[\begin{array}{c} \Delta V' \\ \hdashline \Delta V'' \end{array}\right]^{(\nu+1)} = \left[\begin{array}{c} \Delta P \\ \hdashline \Delta V^2 \end{array}\right]^{(\nu)} \qquad \text{(LF.36)}$$

where

$$\left(\Delta V^2\right)_i^{(\nu)} = V_i^{2*} - V_i^{2(\nu)} \qquad \text{(LF.37)}$$

and the elements of the Jacobian are given by (LF.32), (LF.33) – the partial derivatives of the active power injections - and (LF.38), (LF.39), shown in the following slide (the partial derivatives of the squares of the voltages).

Numerical solution via the NR method

$$J_{VR}:\begin{cases}\dfrac{\partial\left(V_i^2\right)}{\partial V_\ell^{'}}=0\\[2ex]\dfrac{\partial\left(V_i^2\right)}{\partial V_i^{'}}=2V_i^{'}\end{cases}\qquad (LF.38)$$

where

$$J_{VX}:\begin{cases}\dfrac{\partial\left(V_i^2\right)}{\partial V_\ell^{''}}=0\\[2ex]\dfrac{\partial\left(V_i^2\right)}{\partial V_i^{''}}=2V_i^{''}\end{cases}\qquad (LF.39)$$

From the (LF.37) we can note that the two sub-matrices $\partial V^2/\partial V^{'}$ and $\partial V^2/\partial V^{''}$ in (LF.36) have all their elements equal to zero, except for the ones in the main diagonal.

Numerical solution via the NR method

c) Slack bus:

$$\begin{cases} V_s^{'} = V_s^{'} \\ V_s^{''} = 0 \end{cases} \tag{LF.40}$$

To sum up, we obtain:

$$\begin{bmatrix} J_{PR} & J_{PX} \\ J_{QR} & J_{QX} \\ J_{VR} & J_{VX} \end{bmatrix}^{(\nu)} \times \begin{bmatrix} \Delta V^{'} \\ \Delta V^{''} \end{bmatrix}^{(\nu+1)} = \begin{bmatrix} \Delta P \\ \Delta Q \\ \Delta\left(V^2\right) \end{bmatrix}^{(\nu)} \tag{LF.41}$$

Numerical solution via the NR method

Application of the Newton-Raphson method in the Loadflow (Polar):

a) Load buses:

$$\begin{bmatrix} \dfrac{\partial P}{\partial V} & \dfrac{\partial P}{\partial \vartheta} \\ \dfrac{\partial Q}{\partial V} & \dfrac{\partial Q}{\partial \vartheta} \end{bmatrix}^{(\nu)} \times \begin{bmatrix} \Delta V \\ \Delta \vartheta \end{bmatrix}^{(\nu+1)} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}^{(\nu)} \qquad \text{(LF.42)}$$

a) Generator buses:

$$\begin{bmatrix} \dfrac{\partial P}{\partial V} & \dfrac{\partial P}{\partial \vartheta} \end{bmatrix}^{(\nu)} \times \begin{bmatrix} \Delta V \\ \Delta \vartheta \end{bmatrix}^{(\nu+1)} = \left[\Delta P\right]^{(\nu)} \qquad \text{(LF.43)}$$

In the polar coordinate formulation, the number of equations is g and **NOT** $2g$. Indeed, the other g equations that correspond to the voltage magnitudes of the generator buses are not used in the calculation but serve only to define the known parameters. $[\Delta V]$ and $[\Delta \vartheta]$ are the arrays of the variations of the bus voltage magnitudes and phases. The elements of the Jacobian that appear in (LF.42) and (LF.43), calculated based on ◀ are:

Numerical solution via the NR method

$$J_{PV}: \begin{cases} \dfrac{\partial P_i}{\partial V_\ell} = Y_{i\ell} V_i \cos\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \\ \dfrac{\partial P_i}{\partial V_i} = 2 Y_{ii} V_i \cos\gamma_{ii} + \sum_{\substack{\ell=1 \\ \ell\neq i}}^{s} Y_{i\ell} V_\ell \cos\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \end{cases} \qquad (\text{LF.44})$$

$$J_{P\vartheta}: \begin{cases} \dfrac{\partial P_i}{\partial \vartheta_\ell} = Y_{i\ell} V_i V_\ell \sin\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \\ \dfrac{\partial P_i}{\partial \vartheta_i} = -V_i \sum_{\substack{\ell=1 \\ \ell\neq i}}^{s} Y_{i\ell} V_\ell \sin\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \end{cases} \qquad (\text{LF.45})$$

Numerical solution via the NR method

$$J_{QV}:\begin{cases}\dfrac{\partial Q_i}{\partial V_\ell}=Y_{i\ell}V_i\sin\left(\vartheta_i-\vartheta_\ell-\gamma_{i\ell}\right)\\[2ex]\dfrac{\partial Q_i}{\partial V_i}=-2Y_{ii}V_i\sin\gamma_{ii}+\sum\limits_{\substack{\ell=1\\ \ell\neq i}}^{s}Y_{i\ell}V_\ell\sin\left(\vartheta_i-\vartheta_\ell-\gamma_{i\ell}\right)\end{cases}\qquad(\text{LF.46})$$

$$J_{Q\vartheta}:\begin{cases}\dfrac{\partial Q_i}{\partial \vartheta_\ell}=-Y_{i\ell}V_iV_\ell\cos\left(\vartheta_i-\vartheta_\ell-\gamma_{i\ell}\right)\\[2ex]\dfrac{\partial Q_i}{\partial \vartheta_i}=V_i\sum\limits_{\substack{\ell=1\\ \ell\neq i}}^{s}Y_{i\ell}V_\ell\cos\left(\vartheta_i-\vartheta_\ell-\gamma_{i\ell}\right)\end{cases}\qquad(\text{LF.47})$$

Numerical solution via the NR method

To sum up, we obtain:

$$\begin{bmatrix} J_{PV} & J_{P\vartheta} \\ J_{QV} & J_{Q\vartheta} \end{bmatrix}^{(\nu)} \times \begin{bmatrix} \Delta V \\ \Delta\vartheta \end{bmatrix}^{(\nu+1)} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}^{(\nu)} \qquad (\text{LF.48})$$

Numerical solution via the NR method

Application of the Newton-Raphson method in the Loadflow (Mixed):

$$\frac{\partial P_i}{\partial V_\ell} = V_i \left(G_{i\ell} \cos \vartheta_{i\ell} + B_{i\ell} \sin \vartheta_{i\ell} \right)$$

$$\frac{\partial P_i}{\partial V_i} = 2 G_{ii} V_i + \sum_{\ell \neq i} V_\ell \left(G_{i\ell} \cos \vartheta_{i\ell} + B_{i\ell} \sin \vartheta_{i\ell} \right) \quad (LF.49)$$

$$\frac{\partial P_i}{\partial \vartheta_\ell} = V_i V_\ell \left(G_{i\ell} \sin \vartheta_{i\ell} - B_{i\ell} \cos \vartheta_{i\ell} \right)$$

$$\frac{\partial P_i}{\partial \vartheta_i} = -V_i \sum_{\ell \neq i} V_\ell \left(G_{i\ell} \sin \vartheta_{i\ell} - B_{i\ell} \cos \vartheta_{i\ell} \right) \quad (LF.50)$$

Numerical solution via the NR method

Application of the Newton-Raphson method in the Loadflow (Mixed):

$$\frac{\partial Q_i}{\partial V_\ell} = V_i \left(G_{i\ell} \sin \vartheta_{i\ell} - B_{i\ell} \cos \vartheta_{i\ell} \right)$$

$$\frac{\partial Q_i}{\partial V_i} = -2 B_{ii} V_i + \sum_{\ell \neq i} V_\ell \left(G_{i\ell} \sin \vartheta_{i\ell} - B_{i\ell} \cos \vartheta_{i\ell} \right) \qquad (\text{LF.51})$$

$$\frac{\partial Q_i}{\partial \vartheta_\ell} = -V_i V_\ell \left(G_{i\ell} \cos \vartheta_{i\ell} + B_{i\ell} \sin \vartheta_{i\ell} \right)$$

$$\frac{\partial Q_i}{\partial \vartheta_i} = V_i \sum_{\ell \neq i} V_\ell \left(G_{i\ell} \cos \vartheta_{i\ell} + B_{i\ell} \sin \vartheta_{i\ell} \right) \qquad (\text{LF.52})$$

Numerical solution via the NR method

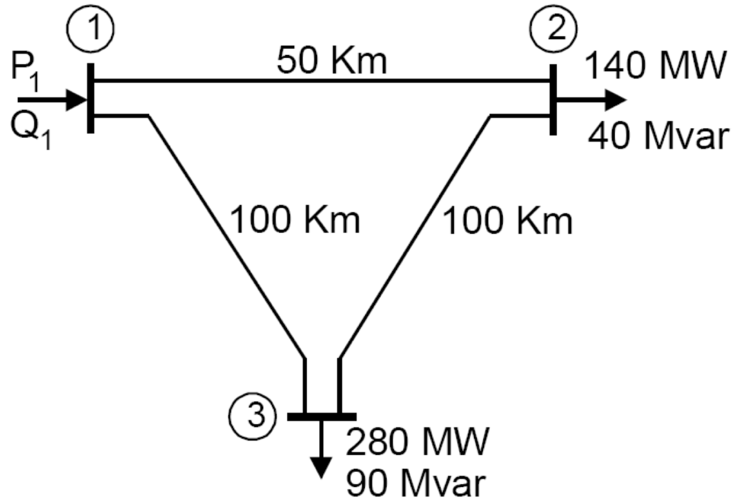


Fig. LF_5: A network of 3 buses

In the network shown in Fig. LF_5, it is given that:

a) Bus 1 receives an active and reactive power injection P_1 and Q_1, to be determined, and the voltage magnitude V_1 is known and is equal to V_1 = 240 kV;

b) Bus 2 distributes active power P_2 = 140 MW and reactive power Q_2 = 40 MVAr;

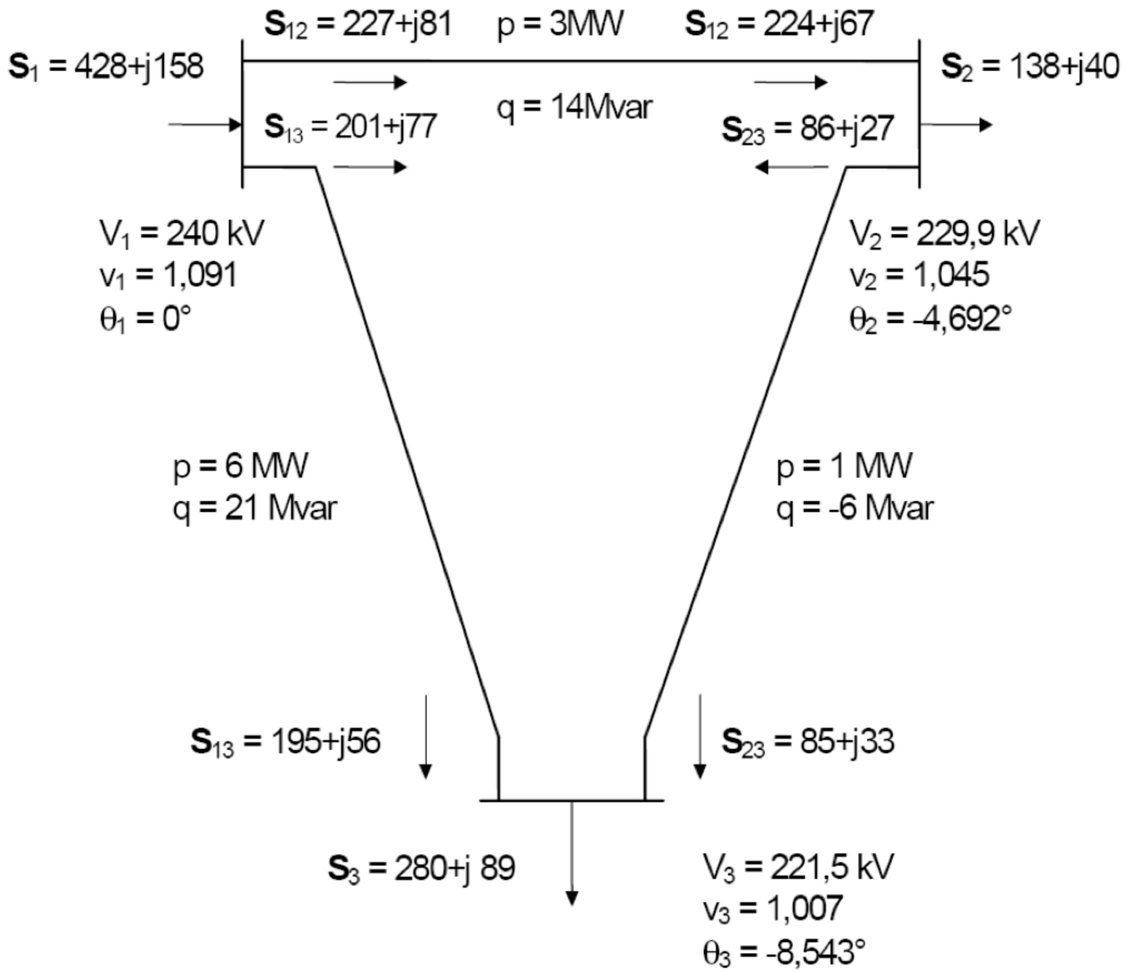
c) Bus 3 distributes active power P_3 = 280 MW and reactive power Q_3 = 90 MVAr;

d) The three lines are overhead lines, their conductors in Al-Ac with diameter equal to 29.3 mm. They are characterized by the same fundamental constant parameters (r = 0.0717 Ω/km, x = 0.424 Ω/km, $b = 2.64 \cdot 10^{-6}$ S/km, $g \cong 0$).

e) The three network buses have nominal voltage equal to 220 kV (V_n = 220 kV).

Numerical solution via the NR method

$$[\mathbf{Y}] = \begin{bmatrix} 5.63 - j33.2 & -3.75 + j22.2 & -1.88 + j11.1 \\ -3.75 + j22.2 & 5.63 - j33.2 & -1.88 + j11.1 \\ -1.88 + j11.1 & -1.88 + j11.1 & 3.75 - j22.2 \end{bmatrix}$$



Numerical solution via the NR method

Iteration 1:

$$\begin{bmatrix} J_{PV} & J_{P\vartheta} \\ J_{QV} & J_{Q\vartheta} \end{bmatrix}^0 = \begin{cases} \dfrac{\partial P_2}{\partial V_2} = 5.2888 & \dfrac{\partial P_2}{\partial V_3} = -1.8767 & \dfrac{\partial P_2}{\partial \vartheta_2} = 35.3110 & \dfrac{\partial P_2}{\partial \vartheta_3} = -11.0977 \\ \dfrac{\partial P_3}{\partial V_2} = -1.8767 & \dfrac{\partial P_3}{\partial V_3} = 3.5827 & \dfrac{\partial P_3}{\partial \vartheta_2} = -11.0977 & \dfrac{\partial P_3}{\partial \vartheta_3} = 23.2044 \\ \dfrac{\partial Q_2}{\partial V_2} = 31.0838 & \dfrac{\partial Q_2}{\partial V_3} = -11.0977 & \dfrac{\partial Q_2}{\partial \vartheta_2} = -5.9712 & \dfrac{\partial Q_2}{\partial \vartheta_3} = 1.8767 \\ \dfrac{\partial Q_3}{\partial V_2} = -11.0977 & \dfrac{\partial Q_3}{\partial V_3} = 32.0288 & \dfrac{\partial Q_3}{\partial \vartheta_2} = 1.8767 & \dfrac{\partial Q_3}{\partial \vartheta_3} = -3.9239 \end{cases}$$

$$\begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}^0 = \begin{bmatrix} -1.4 - P_1 \\ -2.8 - P_2 \\ -0.4 - Q_1 \\ -0.9 - Q_2 \end{bmatrix} = \begin{bmatrix} -1.0588 \\ -2.6294 \\ 1.7136 \\ 0.2367 \end{bmatrix} \qquad \begin{bmatrix} \Delta V \\ \Delta \vartheta \end{bmatrix}^1 = \begin{bmatrix} 0.0522 \\ 0.0119 \\ -0.0848 \\ -0.1515 \end{bmatrix}$$

...